NASA TECHNICAL NOTE

NASA TN D-4430

c. 1

NASA TN D-4430

THE INPUT ADMITTANCE OF A RECTANGULAR APERTURE ANTENNA LOADED WITH A DIELECTRIC PLUG

by Calvin T. Swift and Douglas M. Hatcher

Langley Research Center
Langley Station, Hampton, Va.

NATIONAL AERONAUTICS AND SPACE ADMINISTRATION • WASHINGTON, D. C. • APRIL 1968

THE INPUT ADMITTANCE OF A RECTANGULAR APERTURE ANTENNA LOADED WITH A DIELECTRIC PLUG

By Calvin T. Swift and Douglas M. Hatcher

Langley Research Center
Langley Station, Hampton, Va.

NATIONAL AERONAUTICS AND SPACE ADMINISTRATION

For sale by the Clearinghouse for Federal Scientific and Technical Information
Springfield, Virginia 22151 – CFSTI price $3.00

THE INPUT ADMITTANCE OF A RECTANGULAR APERTURE ANTENNA LOADED WITH A DIELECTRIC PLUG

By Calvin T. Swift and Douglas M. Hatcher
Langley Reaearch Center

SUMMARY

A waveguide-fed rectangular aperture antenna, loaded with a dielectric plug, is considered as a variational problem. The formulation results in an aperture-admittance expression which contains the self and mutual admittances of the transverse electric waveguide modes TE_{01} and TE_{03}. The theoretical results show that a gross distortion of the aperture field and a significant decrease in the aperture admittance occurs when the plug becomes a resonant cavity for the TE_{03} mode.

The input admittance is compared with microwave measurements for the condition in which the external half-space is air. The theory is generally in good agreement with experiment over the entire waveguide band, including the frequency interval around resonance. It is noted that the plug can be used to tune out the susceptance and improve the match of the antenna.

INTRODUCTION

The waveguide-fed rectangular aperture is one of the few antennas for which the input admittance has been analyzed quantitatively as a function of the dielectric constant of the medium over the ground plane. The device is also relatively easy to construct and can be mounted flush to the surface of a flight vehicle. As such, the rectangular aperture antenna seems appealing for possible use in reentry plasma diagnostics.

An air-filled waveguide opening onto the vehicle skin is unsatisfactory because the flow field will be disrupted and hot gases will enter the vehicle through the open waveguide. Some means must therefore be used to seal the aperture internally from the environment. A nonlossy dielectric block, snugly inserted into the aperture with the face flush to the ground plane, is, for practical reasons, an excellent way to isolate the antenna. For example, if the unloaded section of the waveguide is long enough to attenuate modes of higher order, impedance measurements may easily be performed between the plug and the transition from the coaxial line to the waveguide. Also, since the feed radiates into an air-filled guide, problems inherent in matching impedance with dielectric loaded transition from the coaxial line to the waveguide are avoided. However, the aperture may strongly excite standing waves of higher-order waveguide modes in the plug, which may significantly perturb the admittance. The effects of one of these modes, the TE_{03}, is considered as a variational problem. (Examples of the variational approach are given in refs. 1 and 2.) In the formulation, the external medium is assumed to be an infinite half-space of lossy dielectric material; however, calculations are restricted to conditions for which the half-space is air. The results are compared with microwave measurements.

SYMBOLS

A amplitude of TE_{03} mode in the unloaded portion of the waveguide

A_{03} defined in equation (8)

a short dimension of waveguide

b long dimension of waveguide

B susceptance

$C_l(k_x), C_m(k_y)$ modified Hodara functions defined in equation (A2) (l and m are integers)

D nondimensional amplitude of higher-order mode component of aperture electric field

E_x, E_y components of electric field intensity

E_o amplitude of incident wave

E_x^{01} amplitude of TE_{01} component of electric field at the aperture

E_x^{03} amplitude of TE_{03} component of electric field at the aperture

f frequency

$F_1^{lm}(k_x,k_y), F_2(k_x,k_y)$ Fourier transforms defined in equations (A4)

f_1^{lm}, f_2 inverse Fourier transforms defined in equations (A6)

$f(k_x,k_y), g(k_x,k_y)$ Fourier transforms of the potentials ψ and ϕ, respectively

$g(x), h(y)$ integrals defined in equations (A9)

G conductance

H_x, H_y components of magnetic field intensity

H_y^{01} amplitude of TE_{01} component of magnetic field at the aperture

H_y^{03} amplitude of TE_{03} component of magnetic field at the aperture

I reaction integral defined in equation (10)

$j = \sqrt{-1}$

k_x, k_y, k_z three principal continuous mode numbers in region III (fig. 1)

k'_{z01}, k'_{z03} propagation constants in the unloaded section of the waveguide defined in equations (2)

k_{z01}, k_{z03} the z-components of the propagation constants in the loaded section of the waveguide (defined in eqs. (4))

k_o propagation constant of free space

$k = k_r + jk_i$ propagation constant of the medium of region III (fig. 1)

k_r, k_i real and imaginary parts of k

N index of refraction

R, θ, z cylindrical coordinates

t time

V_o electric field amplitude of forward propagating TE_{01} mode in region II

V_{ro} electric field amplitude of backward propagating TE_{01} mode in region II

V_3 electric field amplitude of forward propagating TE_{03} mode in region II

V_4 electric field amplitude of backward propagating TE_{03} mode in region II

x,y,z Cartesian coordinates

Y'_{01}, Y'_{03} characteristic admittance of the TE_{01} and TE_{03} modes, respectively, in region I (defined in eqs. (2))

Y_{01}, Y_{03} characteristic admittance of the TE_{01} and TE_{03} modes, respectively, in region II (defined in eq. (10))

$Y_{ap} = G_{ap} + jB_{ap}$ aperture admittance

$Y_{11} = G_{11} + jB_{11}$ aperture admittance of TE_{01} mode

$Y_{13} = G_{13} + jB_{13}$ mutual admittance between TE_{01} and TE_{03} modes

$Y_{33} = G_{33} + jB_{33}$ aperture admittance of TE_{03} mode

Y_o characteristic admittance of free space, $\sqrt{\frac{\epsilon_o}{\mu_o}}$

y intermediate admittance term defined in equation (37)

y_{ap} aperture admittance divided by Y_{01}

$y_{in} = g_{in} + jb_{in}$ normalized input admittance $\left(\text{aperture admittance divided by } Y'_{01}\right)$

z_o thickness of dielectric prism

Γ reflection coefficient

$\overline{\Gamma}$ reflection coefficient referenced to a short circuit at $z = 0$

Γ_s defined in equation (39)

ϵ_o permittivity of free space

ϵ dielectric constant of the medium of region III

ϵ_r, ϵ_i real and imaginary parts of the dielectric constant in region III

μ_o permeability of free space

ϕ, ψ scalar potentials of region III

ϕ_s phase of the short-circuit reflection coefficient

ω propagating frequency

Subscripts:

l,m indices of Fourier series coefficients

x,y,z components of vectors in the three principal directions

Superscripts:

I fields in the unloaded portion of waveguide $(z < -z_o)$

II fields in the dielectric plug $(-z_o < z < 0)$

III field in the external medium $(z > 0)$

TE denotes transverse electric

TM denotes transverse magnetic

THEORY

The waveguide geometry considered is shown in figure 1. A rectangular waveguide opens onto a flat ground plane, which is covered with a half-space of lossy dielectric material. The waveguide is sealed from the external environment by a nonlossy dielectric prism of thickness z_o, which is inserted so that the face of the plug is flush with the ground plane. The width and height of the plug are assumed to be equal to the inner dimensions of the waveguide.

The dominant TE_{01} mode is incident upon the aperture, where several waveguide modes may be excited by the discontinuity at $z = 0$. However, for conciseness, the aperture electric field is considered as a superposition of only two waveguide modes, specifically, the TE_{01} and TE_{03}. This simplification is equivalent to assuming that (see fig. 1): (1) only those modes which are symmetrical about the origin are excited; (2) the aperture discontinuity induces a negligible E_y component; and (3) all induced modes with cutoff frequencies higher than the TE_{03} mode are negligible.

Under these assumptions, the input admittance, relative to a short circuit placed at $z = 0$, is derived. This derivation is made by imposing variational techniques in connection with the boundary-value problem.

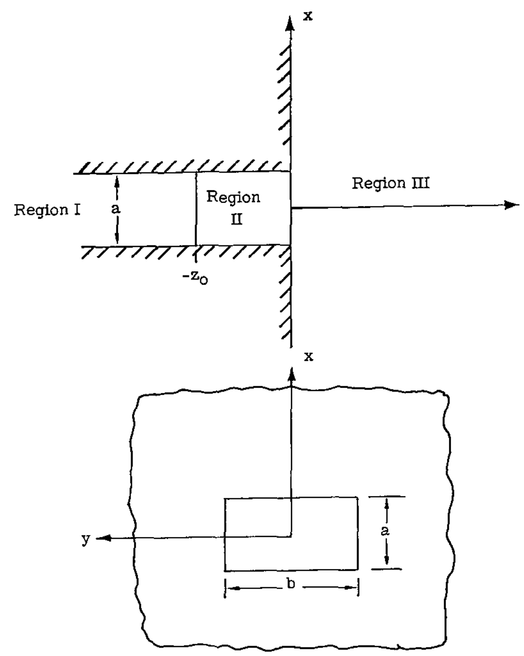


Figure 1.- Waveguide geometry.

The Boundary-Value Problem at $z = -z_o$

The waveguide solutions in region I (fig. 1) consist of an incident TE_{01} mode and reflected TE_{01} and TE_{03} modes. As such, the tangential field components may be written as

$$
\left.\begin{aligned}
E_x^I &= E_o\left(e^{-jk'_{z01}z} + \Gamma e^{jk'_{z01}z}\right)\cos\frac{\pi y}{b} + Ae^{jk'_{z03}z}\cos\frac{3\pi y}{b} \\
E_y^I &= 0 \\
H_x^I &= 0 \\
H_y^I &= Y'_{01}E_o\left(e^{-jk'_{z01}z} - \Gamma e^{jk'_{z01}z}\right)\cos\frac{\pi y}{b} - Y'_{03}Ae^{jk'_{z03}z}\cos\frac{3\pi y}{b}
\end{aligned}\right\} \tag{1}
$$

where a harmonic time dependence of $e^{j\omega t}$ is implicit, and

$$
\left.\begin{aligned}
k'_{z01} &= k_o\sqrt{1 - \left(\frac{\pi}{k_o b}\right)^2} = k_o\frac{Y'_{01}}{Y_o} \\
k'_{z03} &= -jk_o\sqrt{\left(\frac{3\pi}{k_o b}\right)^2 - 1} = k_o\frac{Y'_{03}}{Y_o}
\end{aligned}\right\} \tag{2}
$$

The quantity Γ is the reflection coefficient, and A is the unknown amplitude coefficient of the TE_{03} mode.

In the dielectric plug ($-z_o \leqq z \leqq 0$), a superposition of modal standing waves exists, and the tangential fields are therefore given by

$$\left.\begin{aligned}
E_x^{II} &= \left(V_o e^{-jk_{z01}z} + V_{ro} e^{jk_{z01}z}\right)\cos\frac{\pi y}{b} + \left(V_3 e^{-jk_{z03}z} + V_4 e^{jk_{z03}z}\right)\cos\frac{3\pi y}{b} \\
E_y^{II} &= 0 \\
H_x^{II} &= 0 \\
H_y^{II} &= Y_{01}\left(V_o e^{-jk_{z01}z} - V_{ro} e^{jk_{z01}z}\right)\cos\frac{\pi y}{b} + Y_{03}\left(V_3 e^{-jk_{z03}z} - V_4 e^{jk_{z03}z}\right)\cos\frac{3\pi y}{b}
\end{aligned}\right\} \tag{3}$$

where

$$\left.\begin{aligned}
k_{z01} &= k_o\sqrt{N^2 - \left(\frac{\pi}{k_o b}\right)^2} = k_o\frac{Y_{01}}{Y_o} \\
k_{z03} &= k_o\sqrt{N^2 - \left(\frac{3\pi}{k_o b}\right)^2} = k_o\frac{Y_{03}}{Y_o} \qquad \left(N^2 > \left(\frac{3\pi}{k_o b}\right)^2\right) \\
k_{z03} &= -jk_o\sqrt{\left(\frac{3\pi}{k_o b}\right)^2 - N^2} = k_o\frac{Y_{03}}{Y_o} \qquad \left(N^2 < \left(\frac{3\pi}{k_o b}\right)^2\right)
\end{aligned}\right\} \tag{4}$$

The orthogonality of waveguide modes requires that the corresponding modal values of E_x and H_y be continuous at $z = -z_o$; therefore, equations (1) and (3) immediately lead to the relations

$$\frac{Y'_{01}\left(e^{jk'_{z01}z_o} - \Gamma e^{-jk'_{z01}z_o}\right)}{Y_{01}\left(e^{jk'_{z01}z_o} + \Gamma e^{-jk'_{z01}z_o}\right)} = \frac{\left(V_o e^{jk_{z01}z_o} - V_{ro} e^{-jk_{z01}z_o}\right)}{\left(V_o e^{jk_{z01}z_o} + V_{ro} e^{-jk_{z01}z_o}\right)} \tag{5a}$$

$$Y'_{03} = -\frac{Y_{03}\left(V_3 e^{jk_{z03}z_o} - V_4 e^{-jk_{z03}z_o}\right)}{\left(V_3 e^{jk_{z03}z_o} + V_4 e^{-jk_{z03}z_o}\right)} \tag{5b}$$

Equation (5a) expresses a relationship between the reflection coefficient and the normalized aperture admittance, $y_{ap} = \frac{V_o - V_{ro}}{V_o + V_{ro}}$, which is to be calculated.

Equation (5b) can be manipulated to give

$$V_3 + V_4 = \frac{2V_3}{Y_{03} - Y_{03}'} e^{jk_{z03}z_o} \cos k_{z03}z_o\left(Y_{03} + jY_{03}' \tan k_{z03}z_o\right)$$

$$\equiv D\left(V_o + V_{ro}\right) \tag{6}$$

where D is an unknown coefficient, and

$$V_3 - V_4 = -\frac{2V_3}{Y_{03} - Y_{03}'} e^{jk_{z03}z_o} \cos k_{z03}z_o\left(Y_{03}' + jY_{03} \tan k_{z03}z_o\right)$$

$$\equiv -A_{03}D\left(V_o + V_{ro}\right) \tag{7}$$

The factor A_{03} in equation (7) is defined as

$$A_{03} = \frac{Y_{03}' + jY_{03} \tan k_{z03}z_o}{Y_{03} + jY_{03}' \tan k_{z03}z_o} \tag{8}$$

It therefore follows that the tangential fields at $z = 0$ (eqs. (3)) are given by

$$\left.\begin{aligned}
E_x^{II}(0) &= \left(V_o + V_{ro}\right)\left(\cos \frac{\pi y}{b} + D \cos \frac{3\pi y}{b}\right) \\
E_y^{II}(0) &= H_x^{II}(0) = 0 \\
H_y^{II}(0) &= Y_{01}\left(V_o - V_{ro}\right)\cos \frac{\pi y}{b} - Y_{03}A_{03}D\left(V_o + V_{ro}\right)\cos \frac{3\pi y}{b}
\end{aligned}\right\} \tag{9}$$

The unknown coefficient D and the aperture admittance may now be solved by variational techniques.

The Variational Expression for the Admittance

In order to develop a useful admittance expression, the following reaction integral is considered

$$I = \int_{-a/2}^{a/2} \int_{-b/2}^{b/2} E_x(x,y,0) H_y(x,y,0) dx\, dy \tag{10}$$

This integral has the dimensions of power, but does not contain a complex conjugate of either the electric field intensity or magnetic field intensity in the integrand, which is characteristic of stationary admittance expressions (ref. 3). Substituting equations (9) into equation (10) gives an expression for the integral I in terms of the quantities in region II ($-z_o \leqq z \leqq 0$). The result is

$$I = Y_{01}\left(V_o + V_{ro}\right)^2 \frac{ab}{2}\left(\frac{V_o - V_{ro}}{V_o + V_{ro}} - \frac{Y_{03}}{Y_{01}} A_{03} D^2\right) \tag{11}$$

The expression $\dfrac{V_o - V_{ro}}{V_o + V_{ro}}$ is the normalized aperture admittance y_{ap}. Therefore, equation (11) may be rearranged to give

$$y_{ap} = \frac{2}{ab} \frac{I}{Y_{01}\left(V_o + V_{ro}\right)^2} + \frac{Y_{03}}{Y_{01}} A_{03} D^2 \tag{12}$$

It is assumed that reaction is conserved across the aperture; therefore

$$I = \int_{-a/2}^{a/2} \int_{-b/2}^{b/2} E_x^{II}(x,y,0) H_y^{II}(x,y,0) dx\, dy = \int_{-a/2}^{a/2} \int_{-b/2}^{b/2} E_x^{III}(x,y,0) H_y^{III}(x,y,0) dx\, dy \tag{13}$$

Since the fields in region III consist of continuous modes, it is more convenient to work with the Fourier transforms of E_x and H_y in which case Parseval's theorem (ref. 3, p. 182) may be used to re-express the last integral in equation (13) as follows:

$$\begin{aligned} I &= \int_{-a/2}^{a/2} \int_{-b/2}^{b/2} E_x^{III}(x,y,0) H_y^{III}(x,y,0) dx\, dy \\ &= \frac{1}{(2\pi)^2} \iint_{-\infty}^{\infty} \bar{\bar{E}}_x^{III}\left(k_x,k_y,0\right) \bar{\bar{H}}_y^{III}\left(k_x,k_y,0\right) dk_x\, dk_y \end{aligned} \tag{14}$$

where $\bar{\bar{E}}_x^{III}\left(k_x,k_y,0\right)$ and $\bar{\bar{H}}_y^{III}\left(k_x,k_y,0\right)$ are the double Fourier transforms of the field components; that is

$$\bar{\bar{E}}_x\left(k_x,k_y,0\right) = \iint_{-\infty}^{\infty} E_x(x,y,0) e^{j\left(k_x x + k_y y\right)} dx\, dy \tag{15}$$

The substitution of equation (14) into equation (12) gives

$$y_{ap} = \frac{2}{abY_{01}\left(V_o + V_{ro}\right)^2} \frac{1}{(2\pi)^2} \iint_{-\infty}^{\infty} \bar{\bar{E}}_x^{III}(0)\bar{\bar{H}}_y^{III}(0)dk_x\,dk_y + \frac{Y_{03}}{Y_{01}} A_{03}D^2 \tag{16}$$

The exterior problem ($z > 0$) must now be solved in order to express $\bar{\bar{E}}_x^{III}(0)$ and $\bar{\bar{H}}_y^{III}(0)$ in terms of the unknown coefficient D. When this solution is completed, the stationary character of y_{ap} in equation (16), for which a proof is included in reference 1, may be incorporated to solve for D. This solution is demonstrated in the next section.

The Boundary-Value Problem at $z = 0$

The fields in region III ($z > 0$) are a superposition of continuous modes, and may be derived from TE and TM potential functions, ψ and ϕ, respectively, such that

$$\left.\begin{aligned}
E_x^{III} &= -\frac{\partial\psi}{\partial y} + \frac{1}{j\omega\epsilon}\frac{\partial^2\phi}{\partial x\,\partial z} \\
E_y^{III} &= \frac{\partial\psi}{\partial x} + \frac{1}{j\omega\epsilon}\frac{\partial^2\phi}{\partial y\,\partial z} \\
H_x^{III} &= \frac{1}{j\omega\mu_o}\frac{\partial^2\psi}{\partial x\,\partial z} + \frac{\partial\phi}{\partial y} \\
H_y^{III} &= \frac{1}{j\omega\mu_o}\frac{\partial^2\psi}{\partial y\,\partial z} - \frac{\partial\phi}{\partial x}
\end{aligned}\right\} \tag{17}$$

where the solutions to ψ and ϕ are given by

$$\left.\begin{aligned}
\psi &= \frac{1}{(2\pi)^2}\iint_{-\infty}^{\infty} fe^{-j\left(k_x x + k_y y + k_z z\right)}dk_x\,dk_y \\
\phi &= \frac{1}{(2\pi)^2}\iint_{-\infty}^{\infty} ge^{-j\left(k_x x + k_y y + k_z z\right)}dk_x\,dk_y
\end{aligned}\right\} \tag{18}$$

The mode number k_z is defined by the relationship

$$k_z{}^2 = k^2 - k_x{}^2 - k_y{}^2$$

where k is the complex wave number of the medium of region III. A choice of the branch of k_z must be made so that

$$\left.\begin{aligned} \operatorname{Re} k_z &> 0 \\ \operatorname{Im} k_z &< 0 \end{aligned}\right\} \tag{19}$$

Now, let $\bar{\bar{E}}_x^{III}(z)$, $\bar{\bar{E}}_y^{III}(z)$, $\bar{\bar{H}}_x^{III}(z)$, and $\bar{\bar{H}}_y^{III}(z)$ be the double Fourier transforms of their respective components in physical space; namely,

$$\begin{bmatrix} E_x^{III} \\ E_y^{III} \\ H_x^{III} \\ H_y^{III} \end{bmatrix} = \frac{1}{(2\pi)^2} \iint_{-\infty}^{\infty} \begin{bmatrix} \bar{\bar{E}}_x^{III}(z) \\ \bar{\bar{E}}_y^{III}(z) \\ \bar{\bar{H}}_x^{III}(z) \\ \bar{\bar{H}}_y^{III}(z) \end{bmatrix} e^{-j\left(k_x x + k_y y\right)} dk_x \, dk_y \tag{20}$$

The substitution of equations (20) and (18) into equations (17) results in the following equations:

$$\left.\begin{aligned} \bar{\bar{E}}_x^{III}(z) &= j\left(k_y f + \frac{k_x k_z}{\omega\epsilon} g\right) e^{-jk_z z} \\ \bar{\bar{E}}_y^{III}(z) &= -j\left(k_x f - \frac{k_y k_z}{\omega\epsilon} g\right) e^{-jk_z z} \end{aligned}\right\} \tag{21}$$

At $z = 0$, the tangential components of $\vec{E}$ are continuous across the aperture; that is, $\bar{\bar{E}}_x^{II}(0) = \bar{\bar{E}}_x^{III}(0)$ and $\bar{\bar{E}}_y^{II}(0) = \bar{\bar{E}}_y^{III}(0)$. The modal coefficients f and g may therefore be expressed in terms of the transforms of the tangential components of $\vec{E}^{II}$. The result is

$$\left.\begin{aligned} f &= j \frac{\left(k_x \bar{\bar{E}}_y^{II}(0) - k_y \bar{\bar{E}}_x^{II}(0)\right)}{k_x^2 + k_y^2} \\ g &= -j \frac{\left(k_x \bar{\bar{E}}_x^{II}(0) + k_y \bar{\bar{E}}_y^{II}(0)\right)\omega\epsilon}{k_z\left(k_x^2 + k_y^2\right)} \end{aligned}\right\} \tag{22}$$

From equations (17), (20), and (22), it also follows that

$$\bar{\bar{H}}_y^{III}(0) = \frac{\left(k^2 - k_y^2\right)\bar{\bar{E}}_x^{II}(0) + k_y k_x \bar{\bar{E}}_y^{II}(0)}{\omega \mu_0 k_z} \tag{23}$$

Remaining to be solved are the transforms, $\bar{\bar{E}}_x^{II}(0)$ and $\bar{\bar{E}}_y^{II}(0)$ (eqs. (9)), with substitution of the quantity

$$\bar{\bar{E}}_x^{III}(0)\bar{\bar{H}}_y^{III}(0) \equiv \bar{\bar{E}}_x^{II}(0)\bar{\bar{H}}_y^{III}(0) \tag{24}$$

into the aperture admittance expression.

The transforms $\bar{\bar{E}}_x^{II}(0)$ and $\bar{\bar{E}}_y^{II}(0)$ are given by

$$\left.\begin{aligned}
&\bar{\bar{E}}_y^{II}(0) = 0 \\
&\bar{\bar{E}}_x^{III}(0) = \bar{\bar{E}}_x^{II}(0) = \left(V_o + V_{ro}\right)\int_{-a/2}^{a/2}\int_{-b/2}^{b/2}\left(\cos\frac{\pi y}{b} + D\cos\frac{3\pi y}{b}\right)e^{j\left(k_x x + k_y y\right)}dx\,dy \\
&\qquad \equiv \left(V_o + V_{ro}\right)C_0\left(k_x\right)\left[C_1\left(k_y\right) + DC_3\left(k_y\right)\right]
\end{aligned}\right\} \tag{25}$$

where the C_l's and C_m's are functions which are similar to those defined by Hodara in reference 4 and are explicitly defined as

$$\left.\begin{aligned}
C_0\left(k_x\right) &= \int_{-a/2}^{a/2} e^{jk_x x}dx = \frac{a\sin\left(k_x a/2\right)}{k_x a/2} \\
C_1\left(k_y\right) &= \int_{-b/2}^{b/2}\cos\frac{\pi y}{b}\,e^{jk_y y}dy = \frac{2\pi b\cos\left(k_y b/2\right)}{\pi^2 - \left(k_y b\right)^2} \\
C_3\left(k_y\right) &= \int_{-b/2}^{b/2}\cos\frac{3\pi y}{b}\,e^{jk_y y}dy = \frac{-6\pi b\cos\left(k_y b/2\right)}{(3\pi)^2 - \left(k_y b\right)^2}
\end{aligned}\right\} \tag{26}$$

Substituting equations (25) and (23) into equation (16) yields the aperture admittance expression in the form

$$y_{ap} = \frac{1}{Y_{01}}\frac{2}{\omega\mu_o ab}\frac{1}{(2\pi)^2}\iint_{-\infty}^{\infty}\frac{k^2 - k_y^2}{k_z}C_0\left(k_x\right)C_0\left(k_x\right)\left[C_1\left(k_y\right)C_1\left(k_y\right) + 2DC_1\left(k_y\right)C_3\left(k_y\right)\right.$$

$$\left. + D^2C_3\left(k_y\right)C_3\left(k_y\right)\right]dk_x\,dk_y + \frac{Y_{03}}{Y_{01}}A_{03}D^2 \tag{27}$$

If the following relations are defined

$$Y_{lm} = Y_{ml} = \frac{2}{\omega\mu_o ab}\frac{1}{(2\pi)^2}\iint_{-\infty}^{\infty}\frac{k^2 - k_y^2}{k_z}C_0\left(k_x\right)C_0\left(k_x\right)C_l\left(k_y\right)C_m\left(k_y\right)dk_x\,dk_y$$

$$(l = 1,\ 3;\quad m = 1,\ 3) \tag{28}$$

equation (27) finally reduces to

$$y_{ap} = \frac{Y_{11}}{Y_{01}} + 2D\,\frac{Y_{13}}{Y_{01}} + D^2\left(\frac{Y_{33}}{Y_{01}} + \frac{Y_{03}}{Y_{01}}\,A_{03}\right) \tag{29}$$

Since y_{ap} is stationary with respect to assumed aperture field distributions, it follows that

$$\frac{\delta y_{ap}}{\delta D} = 0 \tag{30}$$

Therefore,

$$D = -\,\frac{Y_{13}/Y_{01}}{\dfrac{Y_{33}}{Y_{01}} + \dfrac{Y_{03}}{Y_{01}}\,A_{03}} \tag{31}$$

At this point, the problem is basically solved. Equation (31) can be substituted into equation (29) to give

$$y_{ap} = \frac{Y_{11}}{Y_{01}} - \frac{\left(Y_{13}/Y_{01}\right)^2}{\dfrac{Y_{33}}{Y_{01}} + \dfrac{Y_{03}}{Y_{01}}\,A_{03}} \tag{32}$$

and the electric field distribution at the aperture is computed by substituting equation (31) into equations (9). This substitution results in

$$E_x^{II}(0) = \left(V_o + V_{ro}\right)\left[\cos\frac{\pi y}{b} - \left(\frac{Y_{13}/Y_{01}}{\dfrac{Y_{33}}{Y_{01}} + \dfrac{Y_{03}}{Y_{01}}\,A_{03}}\right)\cos\frac{3\pi y}{b}\right] \tag{33}$$

If the techniques described by Compton (ref. 1) are used, the integrals Y_{lm} may be manipulated into forms which are convenient for numerical evaluation. These expressions are given in the appendix.

The aperture admittance y_{ap} must now be related to a measurable quantity; namely, the reflection coefficient.

Relationship Between Aperture Admittance and Input Admittance

The input admittance of the antenna relative to a short circuit at the reference plane $z = 0$ is given by

$$y_{in} = \frac{1 - \bar{\Gamma}}{1 + \bar{\Gamma}} \tag{34}$$

with $\bar{\Gamma}$ defined as

$$\bar{\Gamma} = \Gamma e^{-j(\phi_s \pm \pi)} \tag{35}$$

The exponential factor in equation (35) is required in order to account for the phase shift introduced by the dielectric plug when the aperture is short-circuited.

Using the relation $y_{ap} = \dfrac{V_o - V_{ro}}{V_o + V_{ro}}$ in equation (5a) yields

$$\Gamma e^{-2jk'_{z01}z_o} = \frac{1 - y}{1 + y} \tag{36}$$

where

$$y = \frac{Y_{01}}{Y'_{01}} \frac{\left(y_{ap} + j \tan k_{z01}z_o\right)}{\left(1 + jy_{ap} \tan k_{z01}z_o\right)} \tag{37}$$

When a short circuit is placed over the aperture, E goes to zero at $z = 0$; therefore, equations (3) demand that

$$V_{ro,s} = -V_{o,s} \tag{38}$$

where the subscript s denotes the short-circuit values of V_{ro} and V_o.

Equation (5a) therefore gives

$$\Gamma_s e^{-2jk'_{z01}z_o} = e^{j\phi_s} e^{-2jk'_{z01}z_o} = \frac{j \tan k_{z01}z_o - \dfrac{Y_{01}}{Y'_{01}}}{j \tan k_{z01}z_o + \dfrac{Y_{01}}{Y'_{01}}} \tag{39}$$

The substitution of equations (36) and (39) into equation (34) results in

$$y_{in} = \frac{\tan k_{z01}z_o + j \dfrac{Y_{01}}{Y'_{01}} y}{y \tan k_{z01}z_o + j \dfrac{Y_{01}}{Y'_{01}}} \equiv g_{in} + jb_{in} \tag{40}$$

where y is defined in equation (37).

Interpretation of the Theoretical Results

The expression for the aperture admittance has been derived in the form (eq. (32)):

$$y_{ap} = \frac{Y_{11}}{Y_{01}} - \frac{\left(Y_{13}/Y_{01}\right)^2}{\frac{Y_{33}}{Y_{01}} + \frac{Y_{03}}{Y_{01}} A_{03}}$$

where Y_{11}, Y_{13}, and Y_{33} are double integrals which must be solved numerically. In order to attach physical significance to these integrals, consider equations (9) as expressed in the following form

$$\left.\begin{aligned} E_x^{II}(0) &= E_x^{01} \cos \frac{\pi y}{b} + E_x^{03} \cos \frac{3\pi y}{b} \\ H_y^{II}(0) &= H_y^{01} \cos \frac{\pi y}{b} + H_y^{03} \cos \frac{3\pi y}{b} \end{aligned}\right\} \tag{41}$$

where

$$\left.\begin{aligned} E_x^{01} &= V_o + V_{ro} \\ E_x^{03} &= D\left(V_o + V_{ro}\right) \\ H_y^{01} &= Y_{01}\left(V_o - V_{ro}\right) \\ H_y^{03} &= -Y_{03}A_{03}D\left(V_o + V_{ro}\right) \end{aligned}\right\} \tag{42}$$

Since $y_{ap} = \frac{V_o - V_{ro}}{V_o + V_{ro}}$, it follows from equations (42) that equation (32) can be expressed in the form

$$Y_{01}y_{ap} = \frac{H_y^{01}}{E_x^{01}} = Y_{11} - \frac{Y_{13}^2}{Y_{33} - \frac{H_y^{03}}{E_x^{03}}} \tag{43}$$

Equations (31) and (42) also give the relationship

$$\frac{E_x^{03}}{E_x^{01}} = - \frac{Y_{13}}{Y_{33} - \frac{H_y^{03}}{E_x^{03}}} \tag{44}$$

The substitution of equation (44) into equation (43) results in

$$H_y^{01} = E_x^{01} Y_{11} + Y_{13} E_x^{03} \tag{45}$$

and equation (44) can be rewritten as

$$H_y^{03} = E_x^{01}Y_{13} + Y_{33}E_x^{03} \tag{46}$$

In other words, Y_{11} and Y_{33} may be regarded as the self-admittances of the TE_{01} mode and TE_{03} mode, respectively. The integral Y_{13} may be interpreted as the mutual coupling between the modes.

Inspection of equation (32) reveals that the aperture will strongly excite a TE_{03} mode if the mutual admittance is large and if the aperture admittance of the TE_{03} mode is nearly matched to the TE_{03} admittance of the waveguide; that is, if Y_{13}^2 is not small and $Y_{33} \approx -Y_{03}A_{03}$. The numerical results will show that these conditions are satisfied at some point in the waveguide band.

DISCUSSION AND RESULTS

The explicit forms of the admittance integrals Y_{11}/Y_o, Y_{13}/Y_o, and Y_{33}/Y_o, as given in the appendix, were numerically solved in a manner discussed by Compton (ref. 1). As a partial check of the programing, values of the integral Y_{11}/Y_o were compared with the one-mode lossy half-space results of R. C. Rudduck.[1] The numbers differed only in the fourth significant figure, which was considered to be satisfactory agreement.

To the authors' knowledge, the other two integrals have not been computed by previous investigators; therefore, an experimental check appeared to be in order. A commercial RG-49/U C-band waveguide of inner dimensions 4.755 by 2.215 centimeters was selected as an aperture feed; and a slightly undersized rectangular quartz prism, arbitrarily cut to a thickness of 0.75 in. (1.9 cm) was cemented into the guide flush with the aperture to within -0.005 in. (0.13 mm). A standard 3.62-in. (9.2 cm) diameter circular flange was used as a ground plane. It was experimentally ascertained that the flange was large enough to appear as an infinite ground plane, as far as the admittance is concerned.

The admittance integrals were then computed in 200 MHz intervals over the 4.0 GHz to 6.0 GHz band, with air chosen as the external medium. The resultant real and imaginary parts of Y_{13}/Y_o and Y_{33}/Y_o are plotted in figure 2. For reference,

---

[1]Unpublished calculations furnished under NASA grant NsG-448, Ohio State University, 1967.

the quantity $-\frac{Y_{03}A_{03}}{Y_o}$ (eq. (8)) is plotted with Y_{33}/Y_o. The interesting feature of figure 2(b) is that $-\frac{Y_{03}A_{03}}{Y_o}$, which is purely capacitative above cutoff of the TE_{03} mode, cancels the inductive susceptance of Y_{33}/Y_o at a frequency of about 5.44 GHz. Because the real part of Y_{33}/Y_o is small over the band, the second term of the aperture admittance expression (eq. (32))

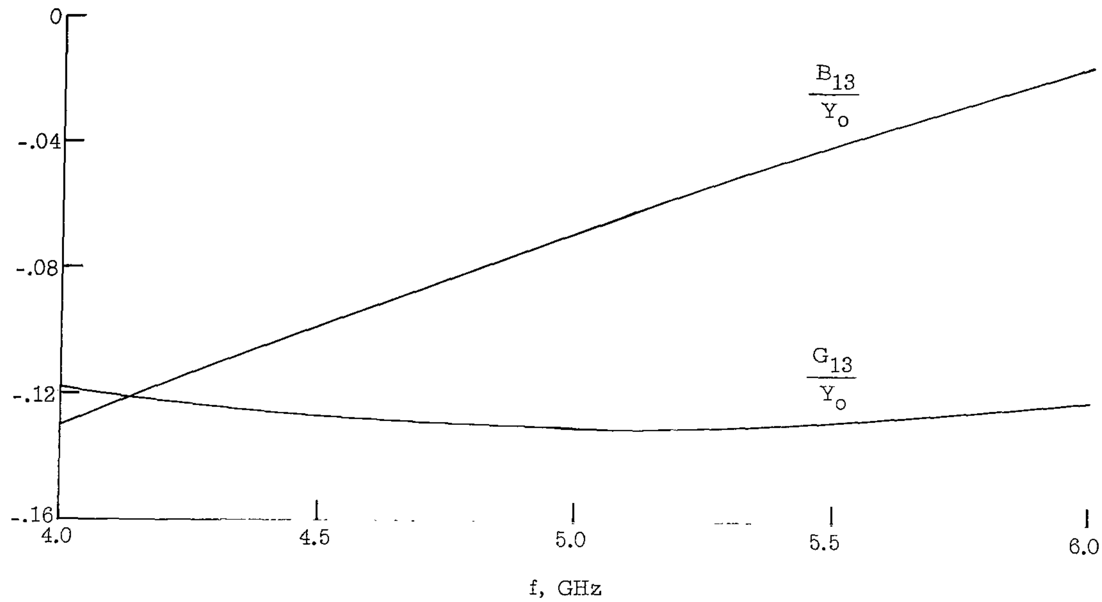
$$y_{ap} = \frac{Y_{11}}{Y_{01}} - \frac{\left(Y_{13}/Y_{01}\right)^2}{\frac{Y_{33}}{Y_{01}} + \frac{Y_{03}}{Y_{01}} A_{03}}$$

becomes large and affects the aperture admittance at frequencies near 5.44 GHz. The frequency interval of the computations was therefore reduced near $f = 5.44$ GHz in order to observe the detailed behavior. The results are given in figure 3, where a sharp drop in both the aperture conductance G_{ap}/Y_o and the aperture susceptance B_{ap}/Y_o is noted near 5.44 GHz. The decrease in G_{ap}/Y_o indicates that power is being coupled out of the TE_{01} mode and appearing as stored energy in the TE_{03} mode. Physically, the plug becomes a resonant cavity for the TE_{03} mode, with a Q of about 200. The integral Y_{11}/Y_o would be the aperture admittance if the TE_{03} mode were not included in the theory, and is plotted for reference in figure 3. From inspection, it is obvious that a one-mode description is an inadequate representation of the aperture admittance near resonance.

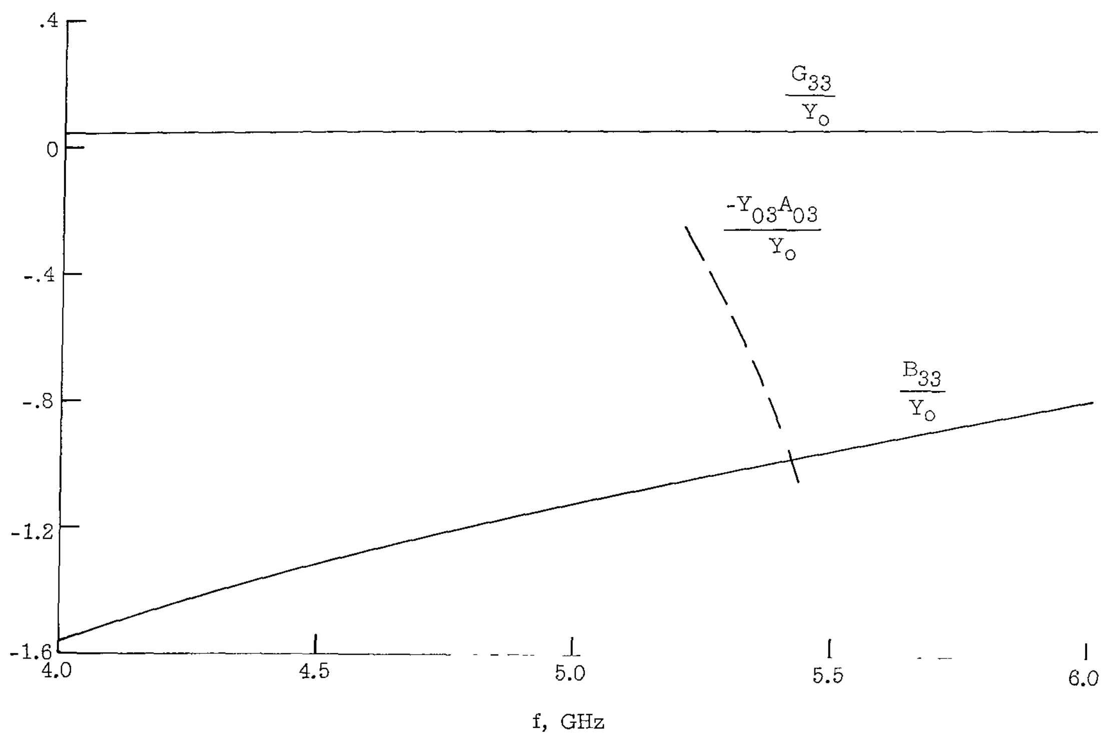
The aperture field distribution $\left|\frac{E_x^{II}(0)}{V_o + V_{ro}}\right|^2$ (eq. (33)) is plotted in figure 4. It is vividly shown in figures 4(a) and 4(b) that substantial changes occur in the electric field distribution as the frequency is swept over a small band about resonance. It is of further interest to note that the TE_{01} and TE_{03} modes add in phase at resonance. If the frequency is significantly above or below resonance, the aperture field distribution does not markedly differ from the $\cos^2 \frac{\pi y}{b}$ TE_{01} mode variation, as shown in figure 4(c).

The input admittance was calculated from equation (40), and the results are shown in figure 5. The experimental data (circles) generally agree with the theory over the entire waveguide band. In particular, the resonant dip predicted by the theory is unmistakably confirmed experimentally. The assumptions made in the theory therefore appear to be justified.

In this particular experiment, the plug introduces a large amount of input capacitance at midband, which diminishes the resonance effects noted in figure 3. This effect may not be obtained in general because the magnitude and frequency location of the perturbation depends upon the index of refraction and thickness of the plug, the waveguide dimensions, and the dielectric constant of the external medium.

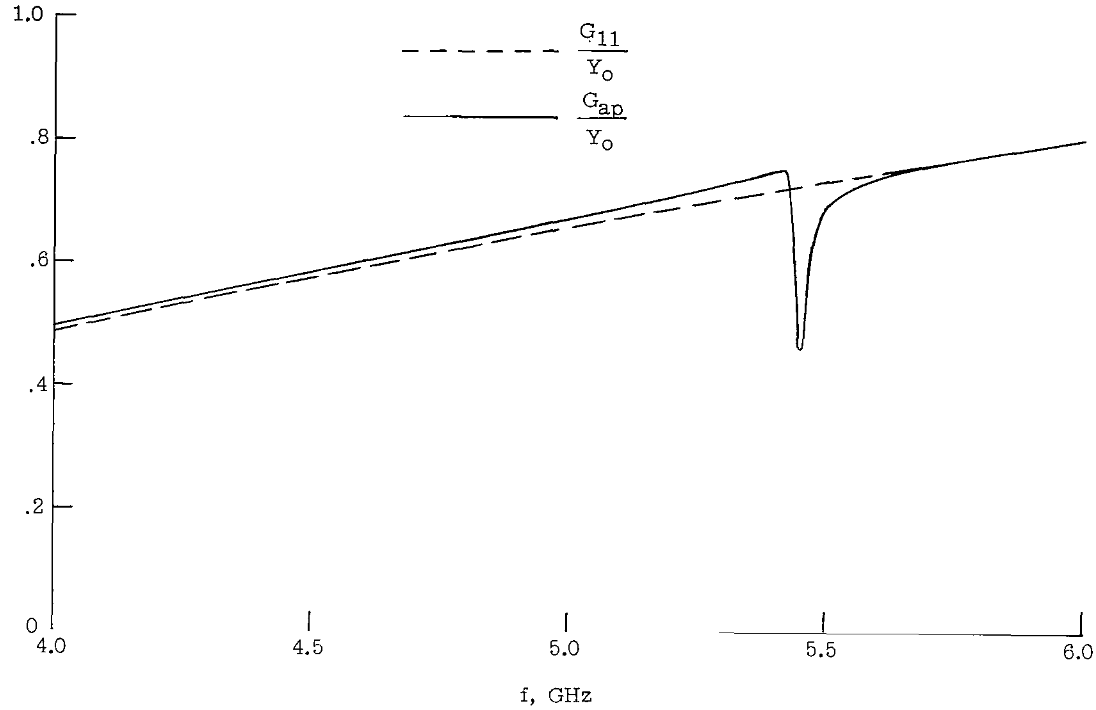


(a) Plot of $\frac{Y_{13}}{Y_0} = \frac{G_{13}}{Y_0} + \frac{jB_{13}}{Y_0}$.

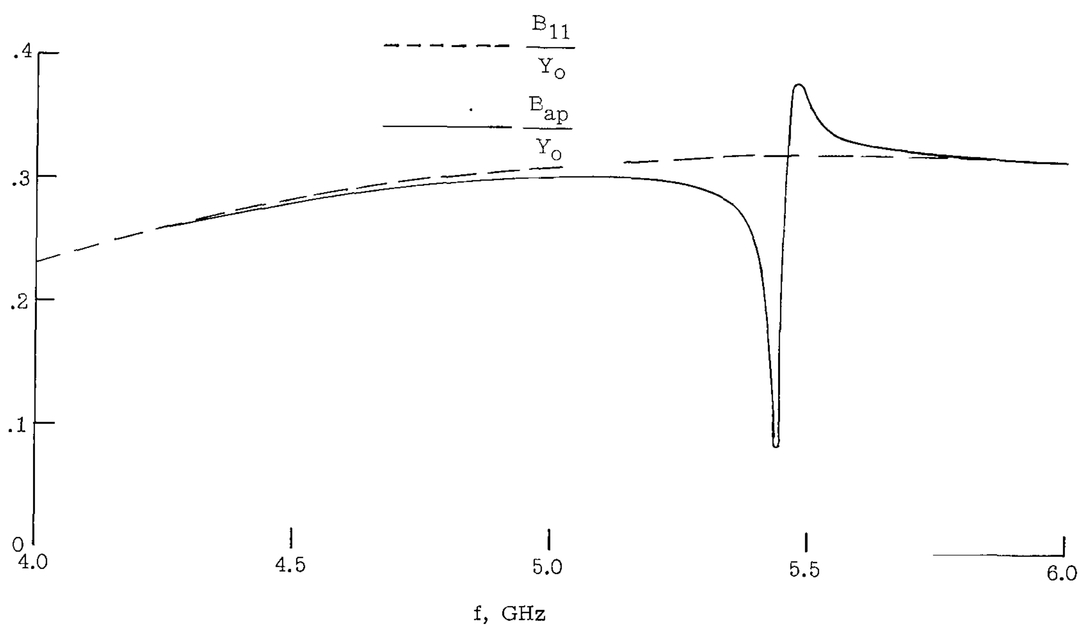


(b) Plot of $\frac{Y_{33}}{Y_0} = \frac{G_{33}}{Y_0} + \frac{jB_{33}}{Y_0}$ and $-\frac{Y_{03}A_{03}}{Y_0}$.

Figure 2.- Plot of the higher-order admittance integrals as a function of frequency.

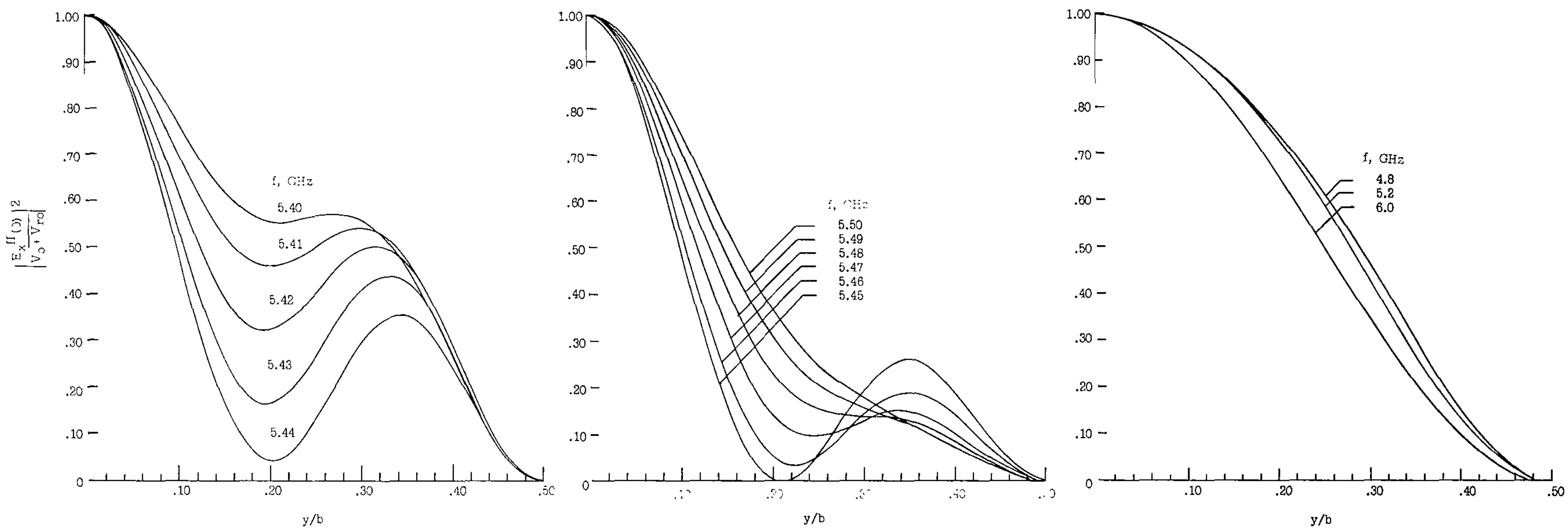


(a) Plots of $\frac{G_{ap}}{Y_o}$ and $\frac{G_{11}}{Y_o}$.



(b) Plots of $\frac{B_{ap}}{Y_o}$ and $\frac{B_{11}}{Y_o}$.

Figure 3.- Plot of the aperture admittance and first-order admittance integral as a function of frequency.

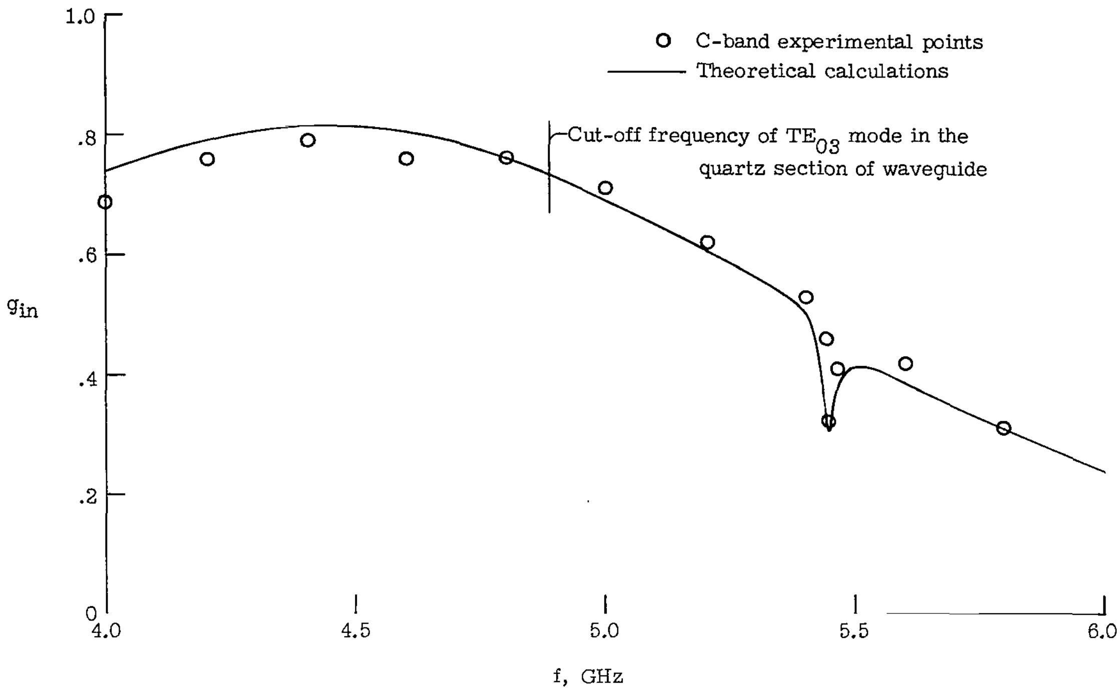


(a) Plots of frequency range below resonance.

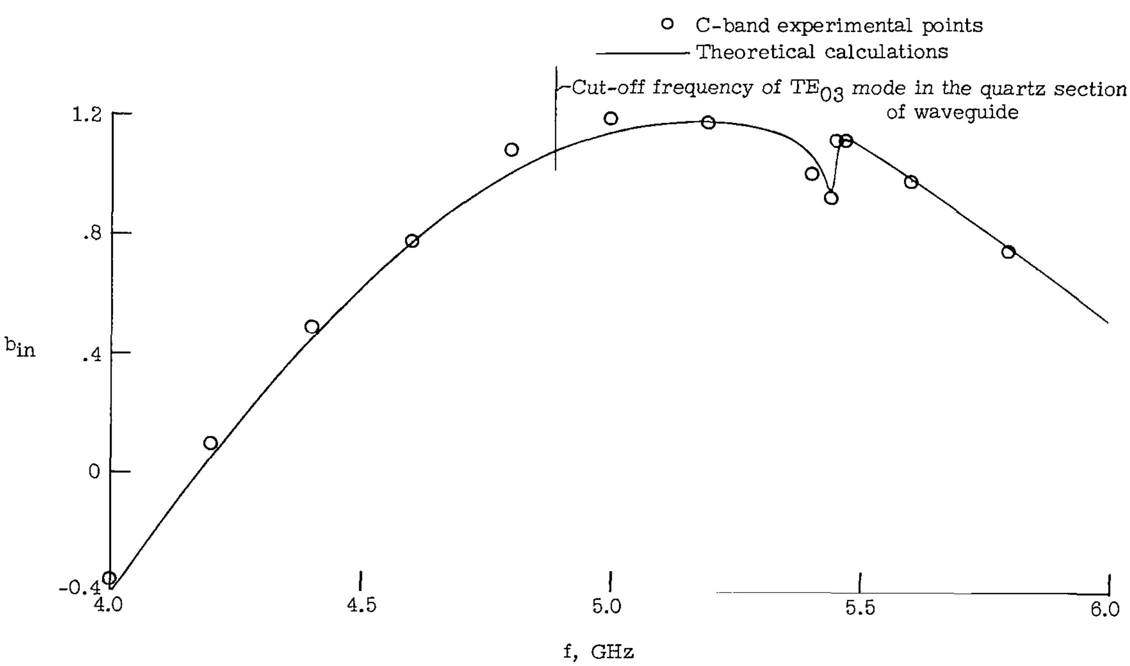
(b) Plots of frequency range above resonance.

(c) Plots for frequencies sufficiently above and below resonance.

Figure 4.- Plots of the aperture electric field distribution across the aperture.



(a) Plot of g_{in} as a function of frequency.



(b) Plot of b_{in} as a function of frequency.

Figure 5.- Experimental and theoretical values of the input admittance as a function of frequency.

If the waveguide is not loaded, the susceptance of this antenna is capacitative over the entire band, and the standing wave ratio at $f = 4.18$ GHz is 1.6. However, when the 0.75-in. (1.9 cm) quartz plug is inserted into the waveguide, the susceptance is tuned out at $f = 4.18$ GHz; and the resultant standing wave ratio is 1.25. Since the tuning point can be changed by choosing an appropriate plug thickness, the free-space match of the antenna can easily be improved at a given frequency in the waveguide band.

Calculations and experiments have been performed at X-band, and similar results were obtained.

CONCLUDING REMARKS

The problem of a dielectric-plug loaded waveguide opening onto a flat conducting ground plane was investigated by variational techniques. The aperture field was assumed to consist of a superposition of the incident TE_{01} mode and an induced TE_{03} mode. Theoretical and experimental results have shown that the dielectric plug can become a resonant cavity for the TE_{03} mode at a discrete frequency in the waveguide band. The coupling of stored energy into the cavity causes distortions in the electric field distribution at the aperture and perturbs the admittance near the resonant frequency. The results show that the susceptance of the antenna can be tuned out to improve the match by adjusting the thickness of the plug.

Langley Research Center,
National Aeronautics and Space Administration,
Langley Station, Hampton, Va., October 10, 1967,
129-01-03-03-23.

APPENDIX

REDUCTION OF THE ADMITTANCE INTEGRALS

If methods described by Compton (ref. 1) are used, the admittance integrals Y_{lm} may be manipulated into forms which are suitable to numerical evaluation. It was shown in equation (28) that

$$Y_{lm} = Y_{ml} = \frac{2}{\omega\mu_o ab} \frac{1}{(2\pi)^2} \iint_{-\infty}^{\infty} \frac{k^2 - k_y^2}{k_z} C_0(k_x)C_0(k_x)C_l(k_y)C_m(k_y)dk_x\, dk_y \tag{A1}$$

where, from a generalization of equations (26), the C_l's are given by

$$C_l(k_y) = -\frac{2l\pi b j^{l+1}\cos\left(k_y \frac{b}{2}\right)}{(l\pi)^2 - \left(k_y b\right)^2} \tag{A2}$$

When the explicit forms of the C_l's are substituted into equation (A1), the integral Y_{lm} may be arranged in the following form

$$Y_{lm} = \frac{1}{(2\pi)^2} \iint_{-\infty}^{\infty} F_1^{lm}(k_x,k_y)\, F_2(k_x,k_y)dk_x\, dk_y \tag{A3}$$

where

$$\left.\begin{aligned}
F_1^{lm}(k_x,k_y) &= -j^{l+m}ml \frac{2ab\left(k^2 - k_y^2\right)\sin^2\left(k_x \frac{a}{2}\right)\cos^2\left(k_y \frac{b}{2}\right)}{\omega\mu_o\left(k_x \frac{a}{2}\right)^2\left[\left(k_y b\right)^2 - (l\pi)^2\right]\left[\left(k_y b\right)^2 - (m\pi)^2\right]} \\
F_2(k_x,k_y) &= \frac{4\pi^2}{k_z}
\end{aligned}\right\} \tag{A4}$$

From Parseval's theorem, Y_{lm} may be converted to an integration over physical space such that

$$\begin{aligned}
Y_{lm} &= \frac{1}{(2\pi)^2} \iint_{-\infty}^{\infty} F_1^{lm^*}(k_x,k_y)\, F_2(k_x,k_y)dk_x\, dk_y \\
&= \iint_{-\infty}^{\infty} f_1^{lm^*}(x,y)f_2(x,y)dx\, dy
\end{aligned} \tag{A5}$$

if f_1 and f_2 are the Fourier transforms of F_1 and F_2, respectively; that is

$$\left.\begin{aligned} f_1^{lm}(x,y) &= \frac{1}{(2\pi)^2}\iint_{-\infty}^{\infty} F_1^{lm}\left(k_x,k_y\right)e^{-j\left(k_x x+k_y y\right)}dk_x\,dk_y \\ f_2(x,y) &= \frac{1}{(2\pi)^2}\iint_{-\infty}^{\infty} F_2\left(k_x,k_y\right)e^{-j\left(k_x x+k_y y\right)}dk_x\,dk_y \end{aligned}\right\} \tag{A6}$$

From Harrington (ref. 3), the Fourier transform of $F_2\left(k_x,k_y\right)$ is

$$f_2(x,y) = \iint_{-\infty}^{\infty} \frac{e^{-j\left(k_x x+k_y y\right)}dk_x\,dk_y}{k_z} = \frac{2\pi j e^{-jk\sqrt{x^2+y^2}}}{\sqrt{x^2+y^2}} \tag{A7}$$

The Fourier transform of $F_1\left(k_x,k_y\right)$ may be separated into the form

$$\begin{aligned} f_1^{lm}(x,y) &= \frac{2b}{\pi^2 a\omega\mu_o}\int_{-\infty}^{\infty}\frac{\sin^2\frac{k_x a}{2}}{k_x^2}e^{-jk_x x}dk_x\int_{-\infty}^{\infty}dk_y\left\{\frac{-j^{l+m}ml\left(k^2-k_y^2\right)\cos^2\frac{k_y b}{2}\,e^{-jk_y y}}{\left[\left(k_y b\right)^2-(l\pi)^2\right]\left[\left(k_y b\right)^2-(m\pi)^2\right]}\right\} \\ &\equiv \frac{2b}{\pi^2 a\omega\mu_o}\,g(x)h(y)^{lm} \end{aligned} \tag{A8}$$

These integrals may be evaluated by the calculus of residues. The result is

$$\left.\begin{aligned} g(x) &= \frac{\pi}{2}\left(a-|x|\right) && (|x| < a) \\ g(x) &= 0 && (|x| > a) \\ h(y)^{lm} &= -j^{l+m}\frac{1}{2\pi^2 b}\left\{\frac{\left[k^2-\left(\frac{m\pi}{b}\right)^2\right]l\,\sin\left(\frac{m\pi|y|}{b}\right)-\left[k^2-\left(\frac{l\pi}{b}\right)^2\right]m\,\sin\left(\frac{l\pi|y|}{b}\right)}{l^2-m^2}\right\} \\ & && (|y| < b;\; l \neq m;\; l,m \text{ odd}) \\ h(y)^{lm} &= \frac{1}{4\pi bl}\left\{\frac{1}{\pi}\left[k^2+\left(\frac{l\pi}{b}\right)^2\right]\sin\left(\frac{l\pi|y|}{b}\right)+\frac{l}{b}\left[k^2-\left(\frac{l\pi}{b}\right)^2\right]\left(b-|y|\right)\cos\left(\frac{l\pi|y|}{b}\right)\right\} \\ & && (|y| < b;\; l = m \neq 0;\; l,m \text{ odd}) \\ h(y)^{lm} &= 0 && (|y| > b \text{ for all } l,m) \end{aligned}\right\} \tag{A9}$$

Therefore, the admittance integrals are given by

$$Y_{lm} = j \frac{16b}{\pi a \omega \mu_o} \int_0^a \int_0^b \frac{g(x)h(y)^{lm} e^{-jk\sqrt{x^2+y^2}}}{\sqrt{x^2 + y^2}} \, dx \, dy \tag{A10}$$

where

$$k = k_r - jk_i = k_o\sqrt{\epsilon_r - j\epsilon_i} \tag{A11}$$

The singularity at $x = y = 0$ may be removed by converting to normalized polar coordinates, with $k_o x = R \cos \theta$ and $k_o y = R \sin \theta$. This requires splitting up the integral defined in equation (A10) into the form

$$Y_{lm} = \int_{\theta=0}^{\theta_o} \int_{R=0}^{k_o a/\cos\theta} \ldots + \int_{\theta=\theta_o}^{\pi/2} \int_{R=0}^{k_o b/\sin\theta} \ldots \tag{A12}$$

where $\tan \theta_o = \frac{b}{a}$.

REFERENCES

1. Compton, R. T., Jr.: The Admittance of Aperture Antennas Radiating Into Lossy Media. Rept. 1691-5 (NASA Grant No. NsG-448), Antenna Lab., Ohio State Univ. Res. Found., March 15, 1964.

2. Galejs, Janis: Admittance of a Waveguide Radiating Into Stratified Plasma. AFCRL-63-394, AD 425 301, U.S. Air Force, Aug. 1963.

3. Harrington, Roger F.: Time-Harmonic Electromagnetic Fields. McGraw-Hill Book Co., Inc., 1961.

4. Knop, C. M.; and Hodara, H.; et al.: Final Report – Research Study of Some RAM Antennas. SN 226 (Contract No. NAS1-4623), Nat. Eng. Sci. Co., June 1965.

NASA-Langley, 1968 —— 7 L-5430

National Aeronautics and Space Administration
WASHINGTON, D. C.

OFFICIAL BUSINESS

FIRST CLASS MAIL

POSTAGE AND FEES PAID
NATIONAL AERONAUTICS AND
SPACE ADMINISTRATION

020 001 32 51 3DS 68074 00903
AIR FORCE WEAPONS LABORATORY/AFWL/
KIRTLAND AIR FORCE BASE, NEW MEXICO 8711

[illegible] CHIEF TECH [illegible]
[illegible]

POSTMASTER: If Undeliverable (Section 158 Postal Manual) Do Not Return

"The aeronautical and space activities of the United States shall be conducted so as to contribute . . . to the expansion of human knowledge of phenomena in the atmosphere and space. The Administration shall provide for the widest practicable and appropriate dissemination of information concerning its activities and the results thereof."

—NATIONAL AERONAUTICS AND SPACE ACT OF 1958

NASA SCIENTIFIC AND TECHNICAL PUBLICATIONS

TECHNICAL REPORTS: Scientific and technical information considered important, complete, and a lasting contribution to existing knowledge.

TECHNICAL NOTES: Information less broad in scope but nevertheless of importance as a contribution to existing knowledge.

TECHNICAL MEMORANDUMS: Information receiving limited distribution because of preliminary data, security classification, or other reasons.

CONTRACTOR REPORTS: Scientific and technical information generated under a NASA contract or grant and considered an important contribution to existing knowledge.

TECHNICAL TRANSLATIONS: Information published in a foreign language considered to merit NASA distribution in English.

SPECIAL PUBLICATIONS: Information derived from or of value to NASA activities. Publications include conference proceedings, monographs, data compilations, handbooks, sourcebooks, and special bibliographies.

TECHNOLOGY UTILIZATION PUBLICATIONS: Information on technology used by NASA that may be of particular interest in commercial and other non-aerospace applications. Publications include Tech Briefs, Technology Utilization Reports and Notes, and Technology Surveys.

Details on the availability of these publications may be obtained from:

SCIENTIFIC AND TECHNICAL INFORMATION DIVISION

NATIONAL AERONAUTICS AND SPACE ADMINISTRATION

Washington, D.C. 20546